Solution sketch 5 - Computational Models - Spring 2017

1. (a) Incorrect. For example, define the languages $A = \{0^n1^n \mid n \geq 0\}$ and $B = \{1\}$, both over the alphabet $\Sigma = \{0,1\}$. Define the function $f : \Sigma^* \mapsto \Sigma^*$ as $f(w) = 1$ if $w \in A$ and $f(w) = 0$ if $w \notin A$. Observe that A is a context-free language, so it is also Turing-decidable. Thus, f is a computable function. Also, $w \in A$ if and only if $f(w) = 1$, which is true if and only if $f(w) \in B$. Hence, $A \leq_m B$. Language A is nonregular, but B is regular since it is finite.

   (b) Correct. A is context-free, so $A \in \mathcal{R}$. Let M_A be the TM that decides A. Let $f(x) = \langle M, x\rangle$, be such that M on input x: Run M_A on x and answer the same. Clearly, f is computable. Now, obviously, $x \in A \iff M$ accepts x.

   (c) Correct. Let f be the reduction from A to B and let g be the reduction from B to C. Define $h : \Sigma^* \to \Sigma^*$ such that $h(x) = g(f(x))$. As f and g are computable, h is computable as well (how?). Also, $x \in A$ if and only if $f(x) \in B$ if and only if $g(f(x)) \in C$ if and only if $h(x) \in C$, as desired.

   (d) Incorrect. Let $A = A_{TM}$ and $B = \overline{A_{TM}}$. It is not the case that $A \leq_m B$ since $B \in$co-$\mathcal{RE}$ but $A \notin$co-$\mathcal{RE}$. Similarly, it is not the case that $B \leq_m A$.

2. We show that $EMPTY_{TM} \leq_m USELESS_{TM}$ by the following mapping reduction. We define $f(\langle M\rangle)$ such that if $\langle M\rangle$ is an illegal encoding then f returns an arbitrary NO instance of $EMPTY_{TM}$ and otherwise returns $\langle M, q_a\rangle$.

   Clearly f is computable. In addition, note that q_a is a useless state if and only if $L(M) = \emptyset$. Thus $\langle M\rangle \in EMPTY_{TM} \iff f(\langle M\rangle) = \langle M, q_a\rangle \in USELESS_{TM}$.

3. We show that $A_{TM} \leq_m S_{TM}$ by the following mapping reduction. We define $f(\langle M, w\rangle)$ such that if $\langle M\rangle$ is an illegal encoding then f returns an arbitrary

NO instance of S_{TM} and otherwise returns $\langle M' \rangle$, whereas M' on input x: If $x = $ '01' accept. Otherwise, run M on w and answers accordingly.

The reduction is computable. If M accepts w then $L(M') = \Sigma^*$ so $\langle M' \rangle \in S_{TM}$. If M does not accept w then $L(M') = \{01\}$ so $\langle M' \rangle \notin S_{TM}$ (the case of an illegal encoding is handled in this direction as well).

4. (a) $L \notin RE \cup coRE$.

   $L \notin RE$: We will show $\overline{H_{TM,\varepsilon}} \leq_m L$. Let M_ε be the TM that on input $\langle M \rangle$, M_ε runs M on ε and answers accordingly. The reduction is $f(\langle M \rangle)$ such that if $\langle M \rangle$ is an illegal encoding then f returns an arbitrary YES instance of L, and otherwise returns $\langle M' \rangle$, whereas M' on input x: Run M on ε for $|x|$ steps. If within the $|x|$ steps M halted, M' rejects. Otherwise, M' simulates M_ε on x and answers accordingly.

   The reduction is computable. If M does not halt on ε then $L(M') = L(M_\varepsilon) = A_{TM,\varepsilon}$ and indeed $A_{TM} \leq_m A_{TM,\varepsilon}$ (the case of an illegal encoding is handled in this direction as well). If M does halt on ε then M' necessarily accepts only a finite number of words, so $L(M') \in R$ and $\langle M' \rangle \notin L$.

   $L \notin coRE$: We will show $H_{TM,\varepsilon} \leq_m L$. The reduction is $f(\langle M \rangle)$ such that if $\langle M \rangle$ is an illegal encoding then f returns an arbitrary NO instance of L, and otherwise returns $\langle M' \rangle$, whereas M' on input x: M' simulates M on ε and then simulates M_ε on x and answers according to the second simulation.

   The reduction is computable. If M halts on ε then $L(M') = L(M_\varepsilon) = A_{TM,\varepsilon}$ and indeed $A_{TM} \leq_m A_{TM,\varepsilon}$. If M does not halt on ε then $L(M') = \emptyset$ so $\langle M' \rangle \notin L$ (the case of an illegal encoding is handled in this direction as well).

   (b) $L \in coRE \setminus R$.

   $L \in coRE$: A TM M' that accepts L, on input $\langle M \rangle$: Use a controlled execution of M over all words in Σ^* where the simulation is done on an extra tape. Whenever, some simulation reaches position $|x| + 20$, accept. The correctness is clear.

   $L \notin R$: We will show $\overline{A_{TM}} \leq_m L$. The reduction is $f(\langle M \rangle, w)$ such that if $\langle M \rangle$ is an illegal encoding then f returns an arbitrary YES instance of L, and otherwise returns $\langle M' \rangle$, whereas M' on input x: Check if x is an accepting computational history of the run of M on w. If it is, keep going right forever. Otherwise, reject. The reduction is computable. If M does not accept w then there is no accepting computational history and for every x we reject. We saw that we don't need any space to the right of x to check it, so $\langle M' \rangle \in L$ (the case of an illegal encoding is handled in this direction as well). If M accepts w then there exists x that is an accepting computational history and on that x, M' reaches the position $|x| + 20$. Thus, $\langle M' \rangle \notin L$.

5. Assume to the contrary that f is computable. Consider the following TM M' , that on input $\langle M \rangle$:

   - If $\langle M \rangle$ is an illegal encoding, reject.
   - Compute $f(\langle M \rangle) = k$.
   - If $k = \infty$ reject.
   - Otherwise, let $m = |Q| \cdot |\Gamma|^k \cdot k$.
   - Emulate M on ε for $m+1$ steps. Accept if M halts and reject otherwise.

   We first argue that $L(M') = H_{TM,\varepsilon}$. If $k = \infty$ then clearly, M doesn't halt on ε. Otherwise m is an upper bound on the number of distinct configurations of M on ε. If M doesn't halt within $m + 1$ steps then it repeats at least one configuration more than once, and thus will never halt. Also, note that M' always halts, so $L(M') \in R$, contradicting the fact that $H_{TM,\varepsilon} \notin R$. Hence, f is not computable.

6. Suppose on the contrary that the set of incompressible strings contains an infinite recursively enumerable subset A. Let E be an enumerator for A. Now, let M denote the Turing machine that on input n a positive integer in binary representation, outputs the first string printed by E that has length at least n (since A is infinite there exists such a string). Then, for every positive integer n, $\langle M, n \rangle$ is a description of an incompressible string s_n of length at least n, and the description has length $\log n + |c_M|$, so we have $n \leq K_U(s_n) \leq \log n + |c_M|$, a contradiction for sufficiently large n.