Computational Models - Exercise #3, Spring 2017

Due: May 22, 23:00

1. Let $G = (\{S, A\}, \{a, b, c\}, R, S)$ be a CFG where R contains the rules:

$$\begin{aligned} S &\rightarrow aSc \mid A \\ A &\rightarrow aAb \mid aA \mid a \end{aligned}$$

   What is $L(G)$? Give a formal proof.

2. For each of the following languages over $\Sigma = \{a, b, c\}$, present a diagram representing a PDA (no need for a correctness proof, but do provide an explanation).

   (a) $L_1 = \left\{a^i b^j a^i : i, j \geq 0\right\}$. Write a formal definition of the PDA as well.

   (b) $L_2 = \{w \in \{a, b\}^\star : \#_a(w) \leq \#_b(w) \leq 2\#_a(w)\}$.

   (c) $L_3 = \left\{a^i b^j c^k : i, j, k \geq 0,\ |i - k| = j\right\}$.

3. For each of the following languages, present a formal definition of a CFG (no need for a correctness proof, but do provide an explanation).

   (a) $L_4 = \{w \in \Sigma^\star : \#_a(w) = 2\#_b(w)\}$ over $\Sigma = \{a, b\}$. Is your grammar ambiguous?

   (b) $L_5 = \{x\$y : x, y \in \{a, b\}^\star \wedge |x| \neq |y|\}$ over $\Sigma = \{a, b, \$\}$. Transform your grammar to Chomsky Normal Form.

4. Prove using the Pumping Lemma that $L_6 = \left\{a^i b^j c^k : 10 \leq i \leq j \leq k\right\}$ over $\Sigma = \{a, b, c, d\}$ is not context-free.

5. Given a DFA A, give a formal construction of a PDA with **three states** M such that $L(A) = L(M)$. Prove correctness.

6. Given L_1 and L_2 over some alphabet Σ, define

$$Even(L_1, L_2) \;=\; \{uv : u \in L_1 \wedge v \in L_2 \wedge |u| = |v|\}.$$

   Prove/disprove:

   (a) If L_1 and L_2 are regular then $Even(L_1, L_2)$ is also regular.

   (b) If L_1 is regular and L_2 is context-free then $Even(L_1, L_2)$ is context-free.

   (c) (not for submission) If L_1 and L_2 are regular then $Even(L_1, L_2)$ is context-free.

7. For an ordered alphabet Σ and $w \in \Sigma^\star$ we denote by $sort(w)$ the word that is obtained from w by sorting its characters in an ascending order. For example, for $\Sigma = \{a, b, c\}$ where $a < b < c$, $sort(caabcc) = aabccc$. Also, $sort(\epsilon) = \epsilon$. Given any $L \subseteq \Sigma^\star$, we define $Sort(L) = \{sort(w) : w \in L\}$. Prove/disprove:

(a) If L is regular over $\Sigma = \{a, b\}$ (where $a < b$) then $Sort(L)$ is also regular.

(b) If L is regular over $\Sigma = \{a, b\}$ (where $a < b$) then $Sort(L)$ is context-free.

(c) If L is regular over $\Sigma = \{a, b, c\}$ (where $a < b < c$) then $Sort(L)$ is context-free.

8. Note that an algorithm is a process that halts on every input and returns the correct answer.

   (a) Show an algorithm that given a PDA M decides whether there exists $w \in L(M)$ for which there exists a decomposition $w = uvxyz$ that satisfies $|vy| \geq 1$ and $uv^ixy^iz \in L(M)$ for every $i \geq 0$.

   (b) Show an algorithm that given a CFG G decides whether $|L(G)| = 2017$.