Exercise 2 - Computational Models - Spring 2017

Note1: We denote by $\#_\sigma(w)$ the number of times the word $\sigma \in \Sigma^*$ is a substring in the word $w \in \Sigma^*$.

Note2: You may freely use results from the lectures and recitations.

1. Determine whether the following languages are regular. Prove your answer.
   (a) $L_1 = \{w \ : \ \#_a(w) \geq \#_b(w)\}$ over $\Sigma = \{a, b, c\}$.
   (b) $L_2 = \{w \ : \ |w| \in \mathbb{N}_{even} \ \wedge \ w = w^R\}$ over $\Sigma = \{0, 1\}$.
   (c) $L_3 = \{w \ : \ |w| \in \mathbb{N}_{even} \ \wedge \ w = w^R\}$ over $\Sigma = \{0\}$.
   (d) $L_4 = \{w \ : \ \exists n \in \mathbb{N} \ s.t. \ |w| = n^3\}$ over $\Sigma = \{1\}$.
   (e) $L_5 = \{w \ : \ \forall 0 \leq k \leq |w|, \#_0(w_1 \dots w_k) \geq \#_1(w_1 \dots w_k)\}$ over $\Sigma = \{0, 1\}$.
   (f) $L_6 = \{a = b + c \ : \ a,b,c \ are \ binary \ integers \ s.t \ a = b + c\}$ over $\Sigma = \{0, 1, =, +\}$.
   (g) $L_7 = \{a^{n_a} b^{n_b} c^{n_c} d^{n_d} \ : \ n_a + n_b = n_c + n_d\}$ over $\Sigma = \{a, b, c, d\}$.
2. (a) For each of the following, write a regular expression for $h(L)$:
   i. $L = L((00 \cup 1)^*)$ and the homomorphism $h : \{0, 1\} \to \{a, b\}^*$ s.t. $h(0) = b, h(1) = \varepsilon$
   ii. $L = \{abab, baba\}$ and the homomorphism $h : \{a, b\} \to \{0, 1\}^*$ s.t. $h(a) = 01$ and $h(b) = 11$
   (b) Let $L = L((00 \cup 1)^*)$ and the homomorphism $h : \{a, b\} \to \{0, 1\}^*$ s.t. $h(a) = 01$ and $h(b) = 10$. Write a regular expression for $h^{-1}(L)$. Give a formal proof.
3. Let $L = \{w \ : \ |w| \ mod \ 3 \ = \ 0\}$ over $\Sigma = \{0, 1\}$. Find each equivalence classes of $\sim_L$. Prove your answer.
4. Prove or disprove:

- Let $Inv(L) = \{xyz \ : \ xy^R z \in L\}$. The regular languages are closed under this operation.
- Let L be a regular language and $C_1, \dots C_k$ L's equivalence classes according to the relation $\sim L$. Let $I \subseteq \{1, \dots k\}$ be a group of indices, then $L_I = \bigcup_{i \in I} C_i$ is regular.
- For any homomorphism h, If L is not regular then $h(L)$ is not regular.

5. Reminder - $\Sigma^* / \backsim L$ is quotient set of the equivalence relation L. the set containing all the equivalence classes of Σ^* induced by $\backsim L$. Let $rank(L) = |\Sigma^* / \backsim L|$ - the number of equivalence classes induced by $\backsim L$. Prove or disprove:
   - Let L be a regular language. Then $rank(L) = rank(\overline{L})$
   - Let L_1, L_2 be a regular languages. Then $rank(L_1 \cap L_2) \leq rank(L_1) \cdot rank(L_2)$
   - Let L be a regular language over Σ and let $M = (Q, \Sigma, \delta, q_0, F)$ the DFA for L from the Myhill-Nerode theorem proof. Then, for every $N = (Q', \Sigma', \delta', S, F')$ a NFA such that $L(N) = L$, $|Q| \leq |Q'|$

6. This question deals with algorithmic problems.
   (a) Let A be a DFA with n states. Prove that $\mid L(A) \mid$ is infinite iff $\exists w \in L(A), n < |w| \leq 2n$.
   (b) Describe an algorithm that given a DFA A, decides if $L(A)$ is infinite. (possible hint: use the pumping lemma).
   (c) Describe an algorithm that given a DFA A, decides if $|L(A)| = 9,122,009$.
   (d) Describe an algorithm that given two DFAs A_1 and A_2, decides if $L(A_1) = L(A_2)$.