Computational Models #2

Handout Mode

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Computational Models - Lecture 2

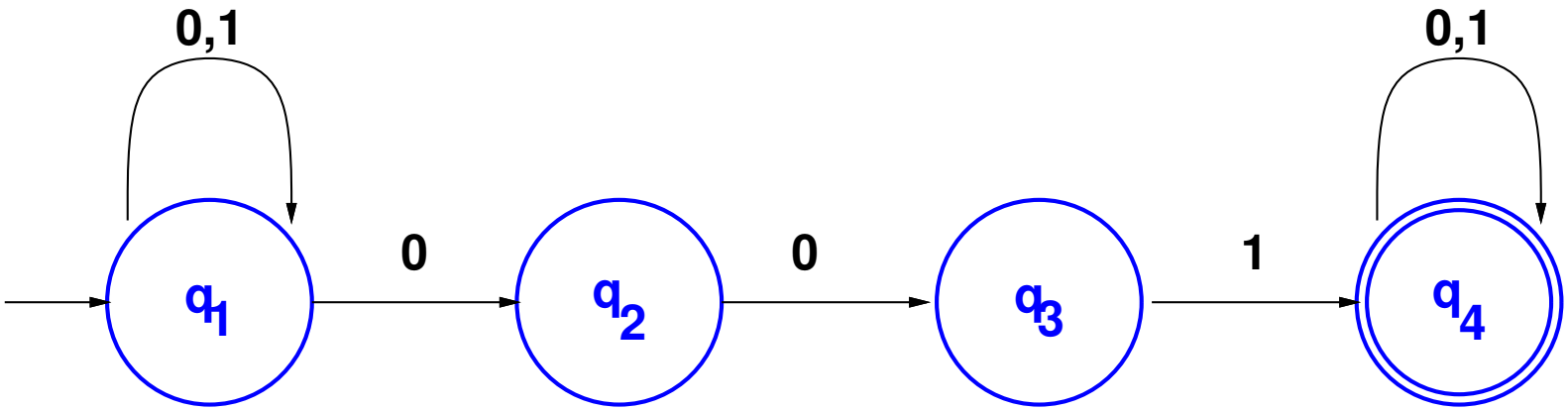
- Nondeterministic Finite Automata (NFA)
- Closure of Regular Languages Under $\bigcup, \circ, ^*$
- Regular expressions
- Equivalence with finite automata

- Sipser's book, 1.1 – 1.3

Part I

Nondeterministic Finite Automata

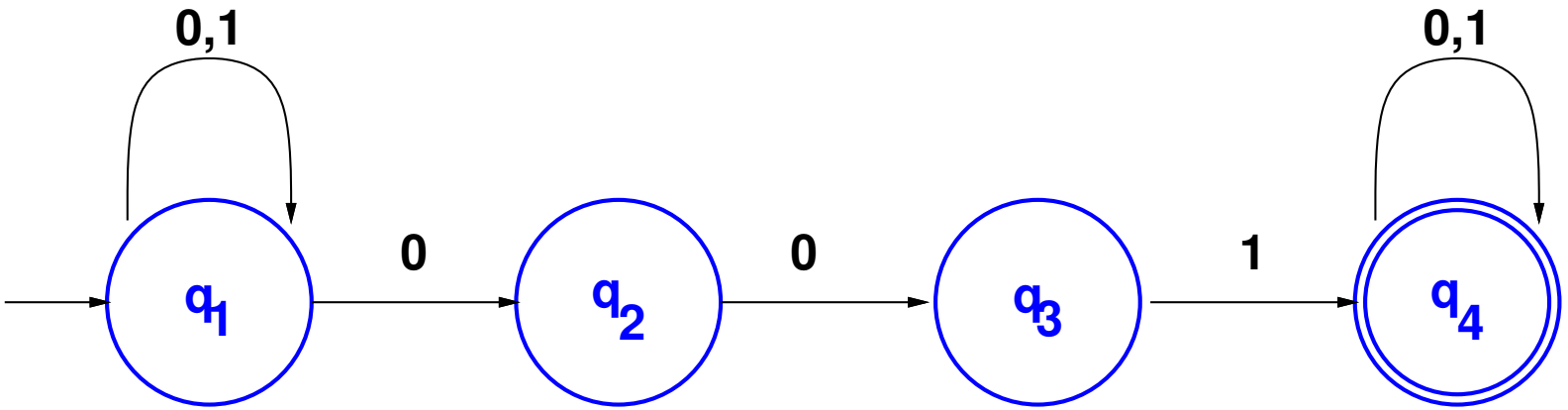
NFA — Nondeterministic Finite Automata



- May have more than one transition labeled with the same symbol,
- May have no transitions labeled with a certain symbol,
- May have transitions labeled with ε, the symbol of the empty string. Will deal with this later

Every DFA "is" also an NFA.

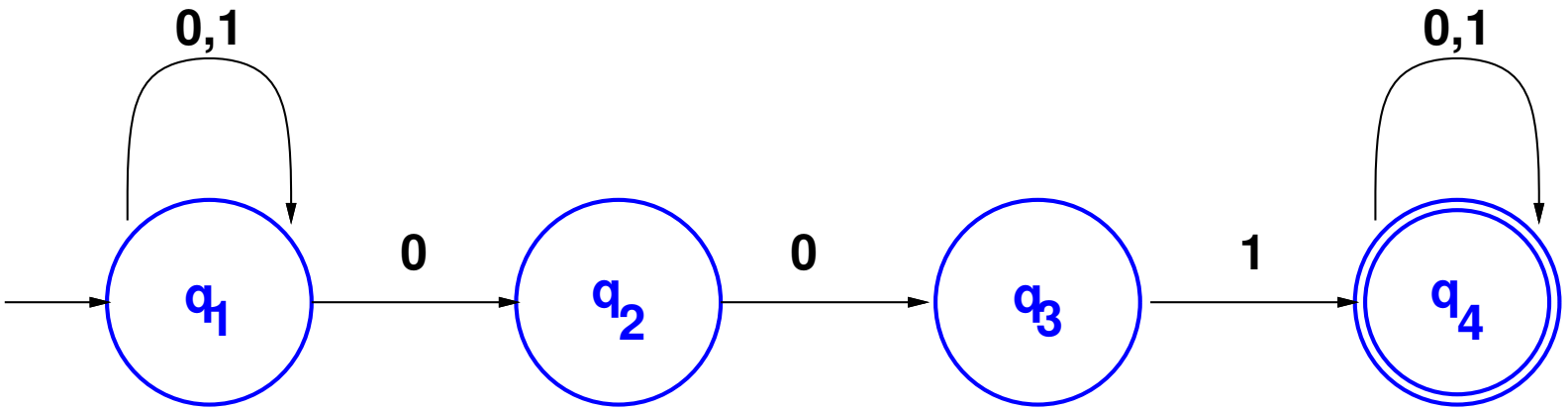
Nondeterministic computation



What happens when more than one transition is possible?

- The machine guesses correctly which path to take.
- If there is an accepting path for the input to come, the machine will take it.
- If there is none, the machine rejects.

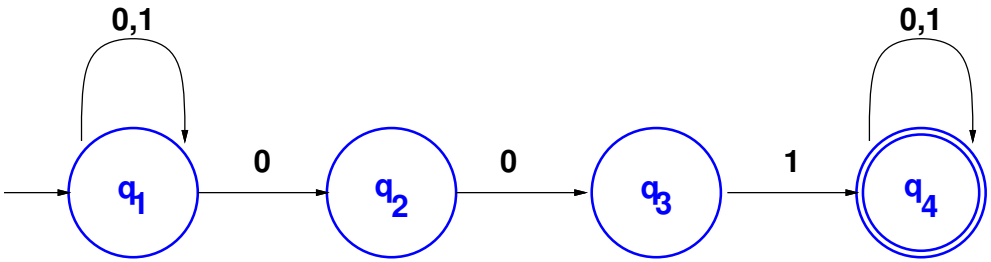
Nondeterministic computation

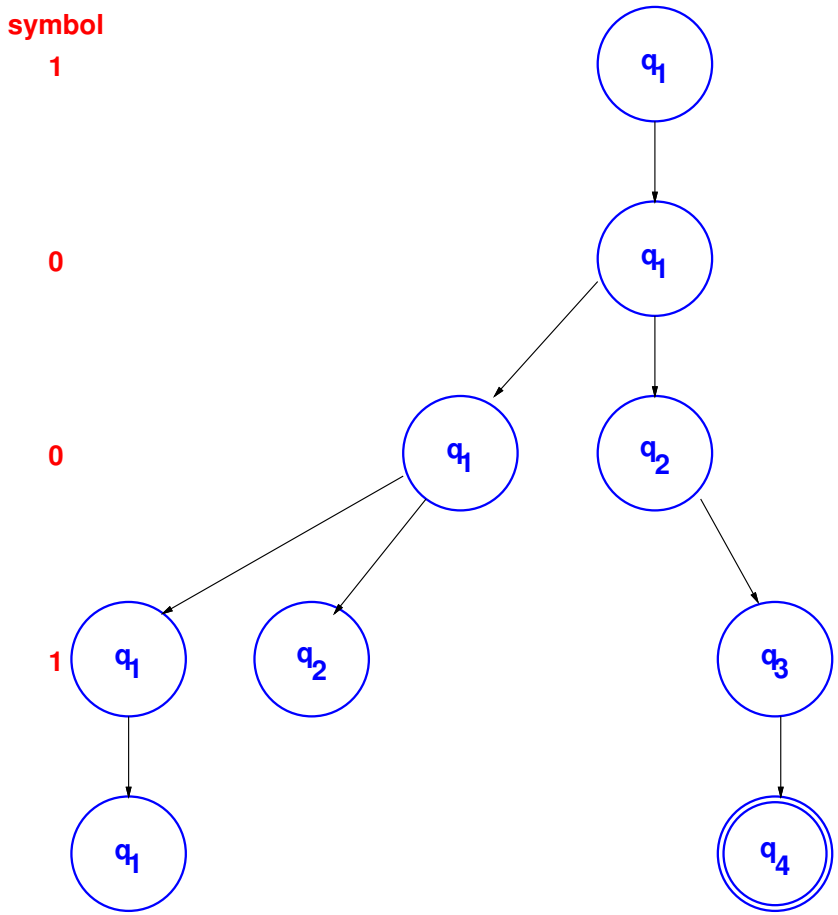


What happens when more than one transition is possible?

- The machine "splits" into multiple copies.
- Each branch follows one possibility.
- Together, branches follow all possibilities.
- If the input doesn't appear, that branch "dies".
- Automaton accepts if some branch accepts.

Computation on 1001





Why nondeterminism?

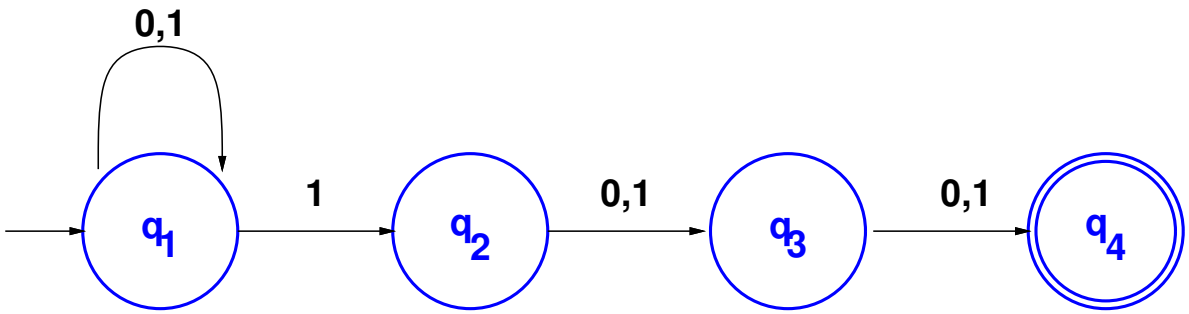
Theorem 1 (Informal, to be proved soon)

Deterministic and nondeterministic finite automata accept exactly the same set of languages.

Q.: So why do we need NFA's?

Design a finite automaton for the language $\mathcal{L}$ — all binary strings with a 1 in their third-to-the-last position?

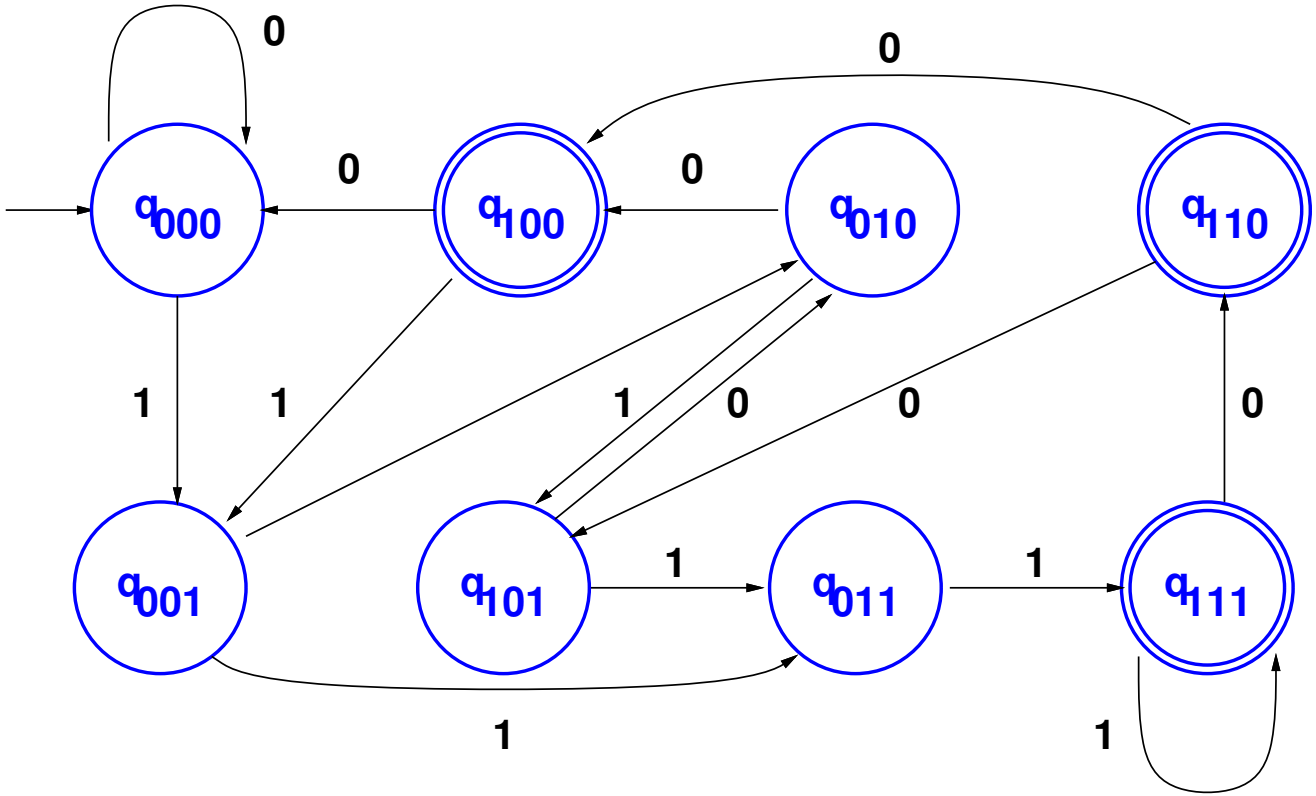
NFA for $\mathcal{L}$



- "Guesses" which symbol is third from the last, and
- checks that indeed it is a 1.
- If guess is premature, that branch "dies", and no harm occurs.

DFA for $\mathcal{L}$

- Have 8 states, encoding the last three observed letters.
- A state for each string in $\{0,1\}^3$.
- Add transitions on modifying the suffix, give the new letter.
- Mark as accepting, the strings $1**$



DFA has few bugs...

DFA – formal definition (reminder)

Definition 2 (DFA)

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \mapsto Q$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Formal model of computation (reminder)

Definition 3

$M = (Q, \Sigma, \delta, q_0, F)$ accepts $w \in \Sigma^*$ if $\widehat{\delta}(q_0, w) \in F$.

Definition 4 ($\widehat{\delta}$)

For DFA $M = (Q, \Sigma, \delta, q_0, F)$, define $\widehat{\delta}: Q \times \Sigma^* \mapsto Q$ by

$$\widehat{\delta}(q, w) = \begin{cases} \delta(\widehat{\delta}(q, w_{1,\ldots,n-1}), w_n), & n = |w| \geq 1 \\ q, & w = \varepsilon. \end{cases}.$$

The language of a DFA (reminder)

Definition 5

The language of a DFA M, denoted $\mathcal{L}(M)$, is the set of strings that M accepts.

Definition 6

A language is called regular, if some deterministic finite automaton accepts it.

NFA – Formal Definition

Let $\mathcal{P}(Q)$ denote the powerset of Q (i.e., all subsets of Q).

Definition 7 (NFA)

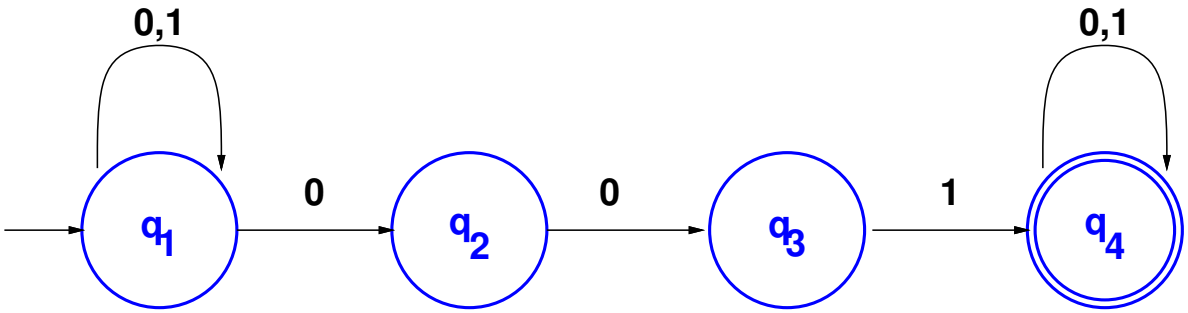
A nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, S, F)$, where

- Q is a finite set called the states
- Σ is a finite set called the alphabet
- $\delta: Q \times \Sigma \mapsto \mathcal{P}(Q)$ is the transition function
- $S \subseteq Q$ is the set of starting states
- $F \subseteq Q$ are the set of accepting states

We sometimes consider an NFA $(Q, \Sigma, \delta, q_0, F)$.

This is merely "syntactic sugar" for the NFA $(Q, \Sigma, \delta, \{q_0\}, F)$

Example



$N_1 = (Q = \{q_1, q_2, q_3, q_4\}, \Sigma = \{0, 1\}, \delta, S = \{q_1\}, F = \{q_4\})$

for δ defined by

	0	1
q_1	$\{q_1, q_2\}$	$\{q_1\}$
q_2	$\{q_3\}$	$\varnothing$
q_3	$\varnothing$	$\{q_4\}$
q_4	$\{q_4\}$	$\{q_4\}$

Not that $\varnothing$ is a valid output for δ

Formal model of computation

Definition 8

$N = (Q, \Sigma, \delta, S, F)$ accepts $w \in \Sigma^*$, if $\widehat{\delta}_N(S, w) \cap F \neq \varnothing$.

Definition 9 ($\widehat{\delta}$)

For NFA $N = (Q, \Sigma, \delta, S, F)$, define $\widehat{\delta}_N \colon P(Q) \times \Sigma^* \mapsto \mathcal{P}(Q)$ by:

$$\text{for } Q' \subseteq Q, \quad \widehat{\delta}_N(Q', w) = \begin{cases} Q', & w = \varepsilon, \\ \bigcup_{q \in \widehat{\delta}_N(Q', w_{1,\ldots,n-1})} \delta(q, w_n), & n = |w| \geq 1. \end{cases}.$$

When clear from he context we will write $\widehat{\delta}$ (i.e., omitting the $_N$).

An equivalent definition

Definition 10 (Equivalent definition)

$N = (Q, \Sigma, \delta, S, F)$ accepts $w = w_1, \ldots, w_n \in \Sigma^n$, if if $\exists r_0, \ldots, r_n \in Q$ s.t.

- $r_0 \in S$
- $r_n \in F$
- $r_{i+1} \in \delta(r_i, w_{i+1})$, for all $0 \leq i < n$.

Equivalence of NFA's and DFA's

Easy: For any DFA M there exists a NFA N such that $\mathcal{L}(N) = \mathcal{L}(M)$.

Other direction is also true.

Theorem 11

For any NFA N there exists a DFA M such that $\mathcal{L}(N) = \mathcal{L}(M)$.

- Given an NFA N, we construct a DFA M, that accepts the same language.
- Make DFA emulates all possible NFA states.
- As consequence of the construction, if the NFA has k states, the DFA has 2^k states (an exponential blow up).

Equivalence of NFA's and DFA's, the DFA

Let $N = (Q, \Sigma, \delta, S, F)$.

Construction 12 ($M = (Q_M, \Sigma, \delta_M, d_0, F_M)$)

- $Q_M = \{[R] : R \subseteq Q\}$.
- $d_0 = [S]$
- $F_M = \{[R] \in Q_M : R \cap F \neq \emptyset\}$
- For $[R] \in Q_M$ and $\sigma \in \Sigma$, let $\delta_M([R], \sigma) = [\widehat{\delta}_N(R, \sigma)]$ $\left(= \left[\bigcup_{r \in R} \delta(r, \sigma)\right]\right)$.

To prove equivalence, we need to prove that

$$\widehat{\delta}_N(S, w) \cap \mathcal{F} \neq \emptyset \iff \widehat{\delta}_M(d_0, w) \in F_M$$

The above is an immediate corollary of the following claim:

Claim 13

$[\widehat{\delta}_N(S, w)] = \widehat{\delta}_M(d_0, w)$ for every $w \in \Sigma^*$.

Proving $[\widehat{\delta}_N(S,w)] = \widehat{\delta}_M(d_0,w)$

The proof is by induction on the length of w.

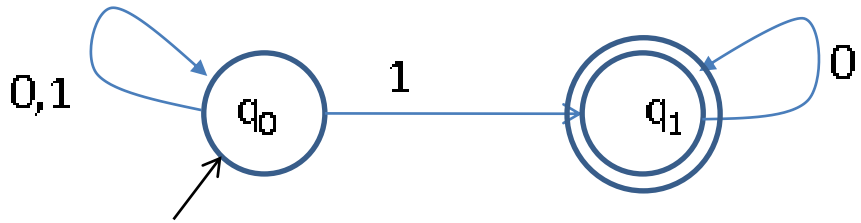
- $|w| = 0$, by definition.
- Assume for words of length $(m-1)$, and let $x = y\sigma$, where y is a word of length $(m-1)$ and $\sigma \in \Sigma$.
- Let $Q_y = \widehat{\delta}_N(S,y)$ and $d_y = \widehat{\delta}_M(d_0,y)$.
- Compute

$$\begin{aligned}
\widehat{\delta}_M(d_0,x) &= \delta_M(d_y,\sigma) && \text{(By definition of } \widehat{\delta}_M\text{)}\\
&= \delta_M([Q_y],\sigma) && \text{(By i.h)}\\
&= [\widehat{\delta}_N(Q_y,\sigma)] && \text{(By definition of } \delta_M\text{)}\\
&= [\bigcup_{q\in Q_y} \delta(q,\sigma)] && \text{(By definition of } \widehat{\delta}_N\text{)}\\
&= [\bigcup_{q\in \widehat{\delta}_N(S,y)} \delta(q,\sigma)]\\
&= [\widehat{\delta}_N(S,x)]. \quad \square && \text{(By definition of } \widehat{\delta}_N\text{)}
\end{aligned}$$

- But how come we encode infinite parallel "threads" into a single thread?

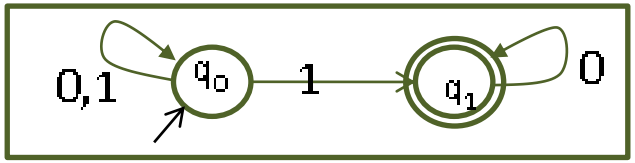
Example: NFA ⇒ DFA

NonDeterministic Automata:



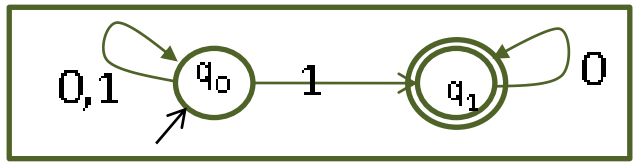
Example: NFA ⇒ DFA

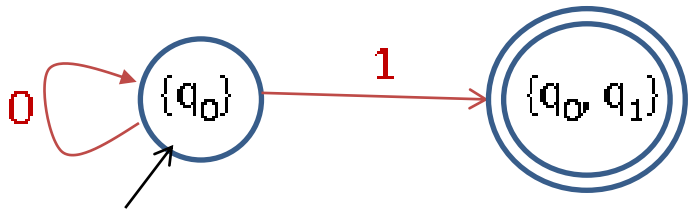
Deterministic automata - set of states:



Example: NFA ⇒ DFA

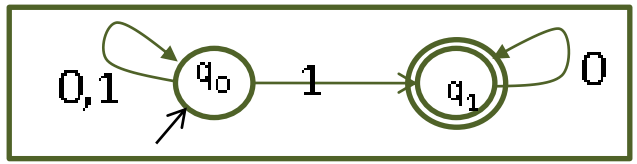
Transitions from $[\{q_0\}]$:

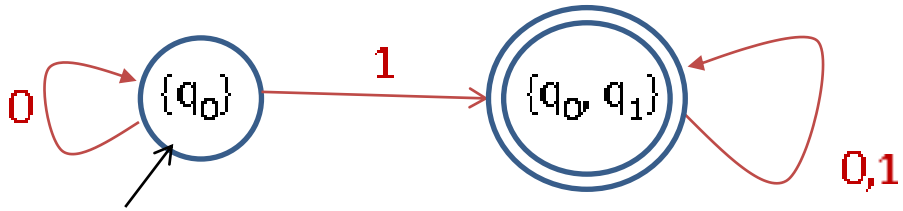




Example: NFA ⇒ DFA

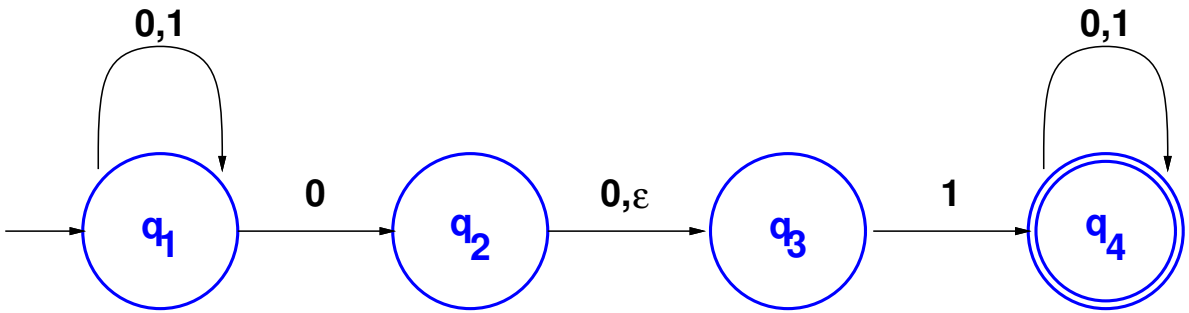
Transitions from $[\{q_0, q_1\}]$:





Transitions from $[\varnothing]$ and $[\{q_1\}]$?

NFA with ε-moves



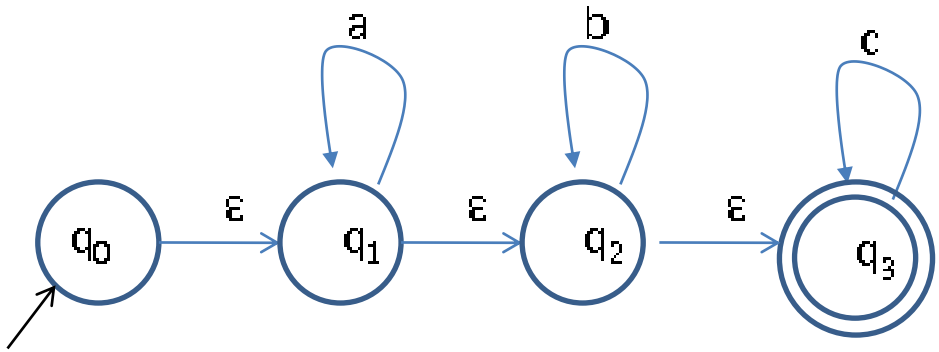
What is the interpretation of ε transitions ?

What will happen with 101 ?

How to distinguish ε (that stands form the empty word) from the ε above (that stands “free transition”)

Example: NFA with ε-moves

$$\mathcal{L} = \{a^i b^j c^k \mid i, j, k \geq 0\}$$



NFA — Formal definition with ε-moves

Transition function δ is going to be different.

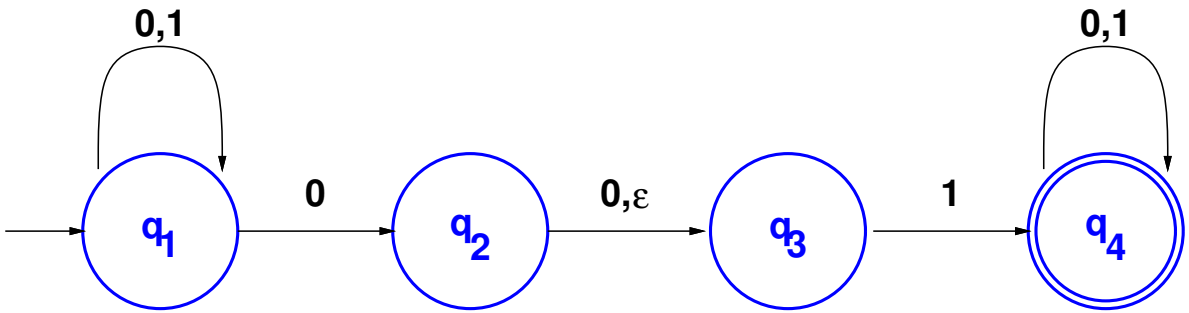
- Let $\mathcal{P}(Q)$ denote the powerset of Q.
- Let Σ_ε denote the set $\Sigma \cup \{\varepsilon\}$.

Definition 14 (NFA, with ε-moves)

A nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, S, F)$:

- Q is a finite set called the states
- Σ is a finite set called the alphabet
- $\delta : Q \times \Sigma_\varepsilon \mapsto \mathcal{P}(Q)$ is the transition function
- $S \subseteq Q$ is the set of starting state
- $F \subseteq Q$ is the set of accepting states

Example



$N_1 = (Q, \Sigma, \delta, S, F)$:

- $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0, 1\}$, $S = \{q_1\}$ and $F = \{q_4\}$.
- δ is

	0	1	ε
q_1	$\{q_1, q_2\}$	$\{q_1\}$	$\varnothing$
q_2	$\{q_3\}$	$\varnothing$	$\{q_3\}$
q_3	$\varnothing$	$\{q_4\}$	$\varnothing$
q_4	$\{q_4\}$	$\{q_4\}$	$\varnothing$

Formal model of computation, with ε-moves

Definition 15

$N = (Q, \Sigma, \delta, S, F)$ accepts $w \in \Sigma^*$, if $\widehat{\delta}_N(S, w) \cap F \neq \varnothing$.

Definition 16

For NFA $N = (Q, \Sigma, \delta, S, F)$, let
$E(q) = \{q' \in Q$: q' can be reached from q by 0 or more ε transitions$\}$
(i.e., $\{q'$: $\exists q_1, \ldots, q_k \in Q$ s.t. $q_1 = q \wedge q_k = q' \wedge \forall i \in [k-1]\ q_{i+1} \in \delta(q_i, \varepsilon)\}$)

$E(Q') = \bigcup_{q \in Q'} E(q)$.

Q: is it always the case that $q \in E(q)$? Yes

Definition 17 ($\widehat{\delta}$)

For NFA $N = (Q, \Sigma, \delta, S, F)$, define $\widehat{\delta}_N$: $P(Q) \times \Sigma^* \mapsto \mathcal{P}(Q)$ by:

for $Q' \subseteq Q$, $\widehat{\delta}_N(Q', w) = \begin{cases} E(Q'), & w = \varepsilon, \\ E\left(\bigcup_{r \in \widehat{\delta}(Q', w_1, \ldots, w_{n-1})} \delta(r, w_n)\right), & n = |w| \geq 1. \end{cases}$.

When does N accept the empty string?

An equivalent definition

For $a \in (\Sigma_\varepsilon)^*$, let $d(a) \in \Sigma^*$ be a without the ε symbols.

Example: $d(\varepsilon 01 \varepsilon\varepsilon 3\varepsilon) = 013$

Definition 18 (Equivalent definition)

$N = (Q, \Sigma, \delta, S, F)$ accepts $w \in \Sigma^*$, if exist $a = (a_1 a_2 \dots a_k) \in (\Sigma_\varepsilon)^k$ and $r_0, \dots, r_k \in Q$ s.t.

- $w = d(a)$.
- $r_0 \in S$
- $r_k \in F$
- $r_{i+1} \in \delta(r_i, a_{i+1})$, for all $0 \le i < k$.

Removing ε-transitions

Given NFA $N = (Q, \Sigma, \delta, S, F)$ with ε-transitions, we create an equivalent NFA $N' = (Q, \Sigma, \delta', S', F)$ with no ε-transitions.

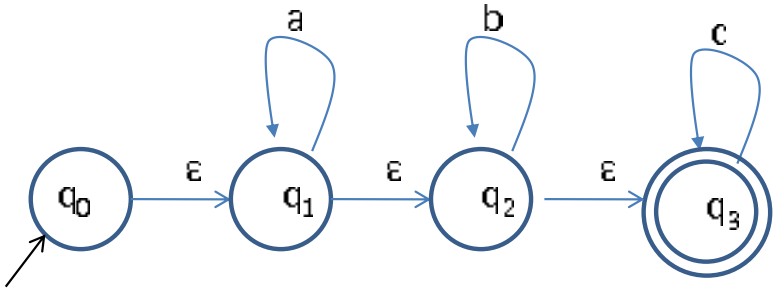
- $S' = E(S)$
- $\delta'(q, a) = E(\delta(q, a))$

It is not hard to prove that $\widehat{\delta}_N(S, w) = \widehat{\delta}_{N'}(S', w)$ for any $w \in \Sigma^*$.

Thus, $\mathcal{L}(N) = \mathcal{L}(N')$.

Example: Removing ε-transitions

NonDeterministic Automata with ε-transitions



The nonDeterministic automata without ε-transitions

- $S' = \{q_0, q_1, q_2, q_3\}$
- δ' is

	a	b	c
q_0	$\varnothing$	$\varnothing$	$\varnothing$
q_1	$\{q_1, q_2, q_3\}$	$\varnothing$	$\varnothing$
q_2	$\varnothing$	$\{q_2, q_3\}$	$\varnothing$
q_3	$\varnothing$	$\varnothing$	$\{q_3\}$

Part II

Closure of Regular Languages, Revisited

Regular languages, revisited

By definition, a language is regular if it is accepted by some DFA.

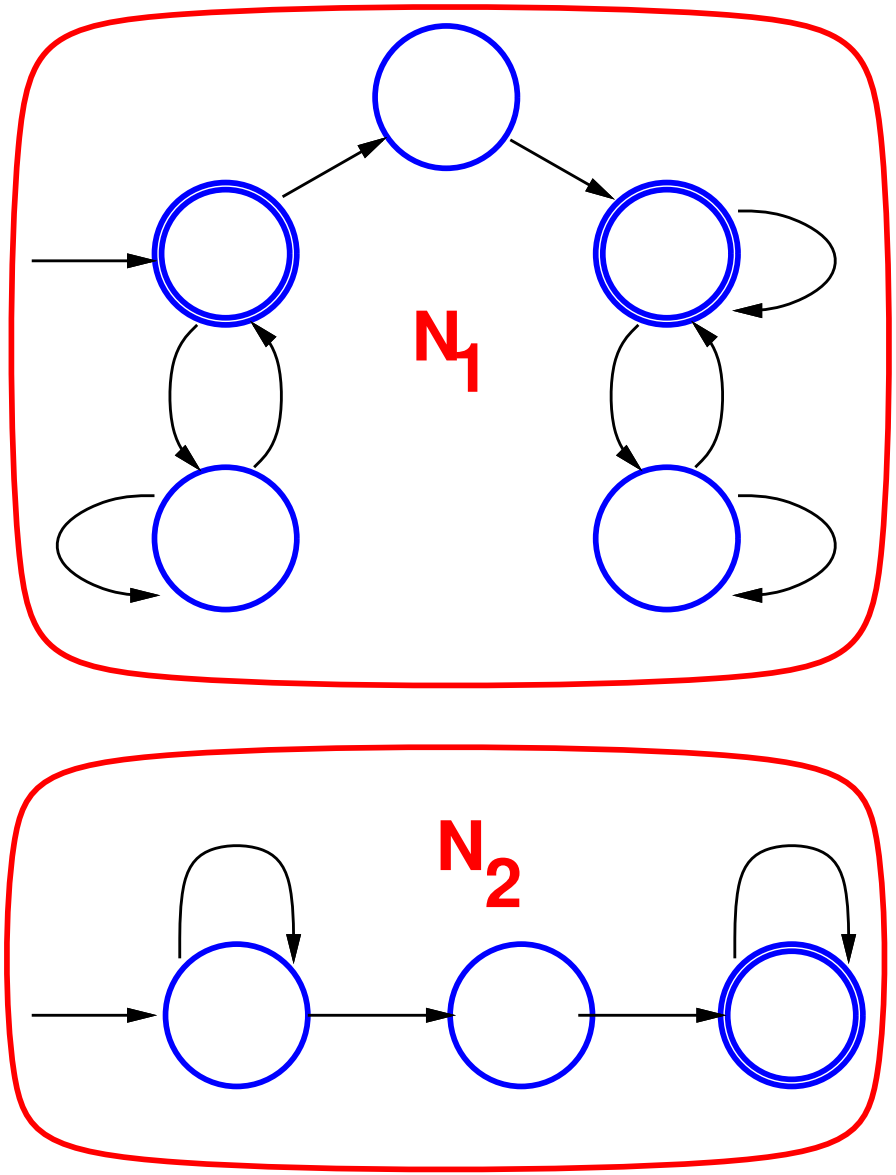
Corollary 19

A language is regular if and only if it is accepted by some NFA.

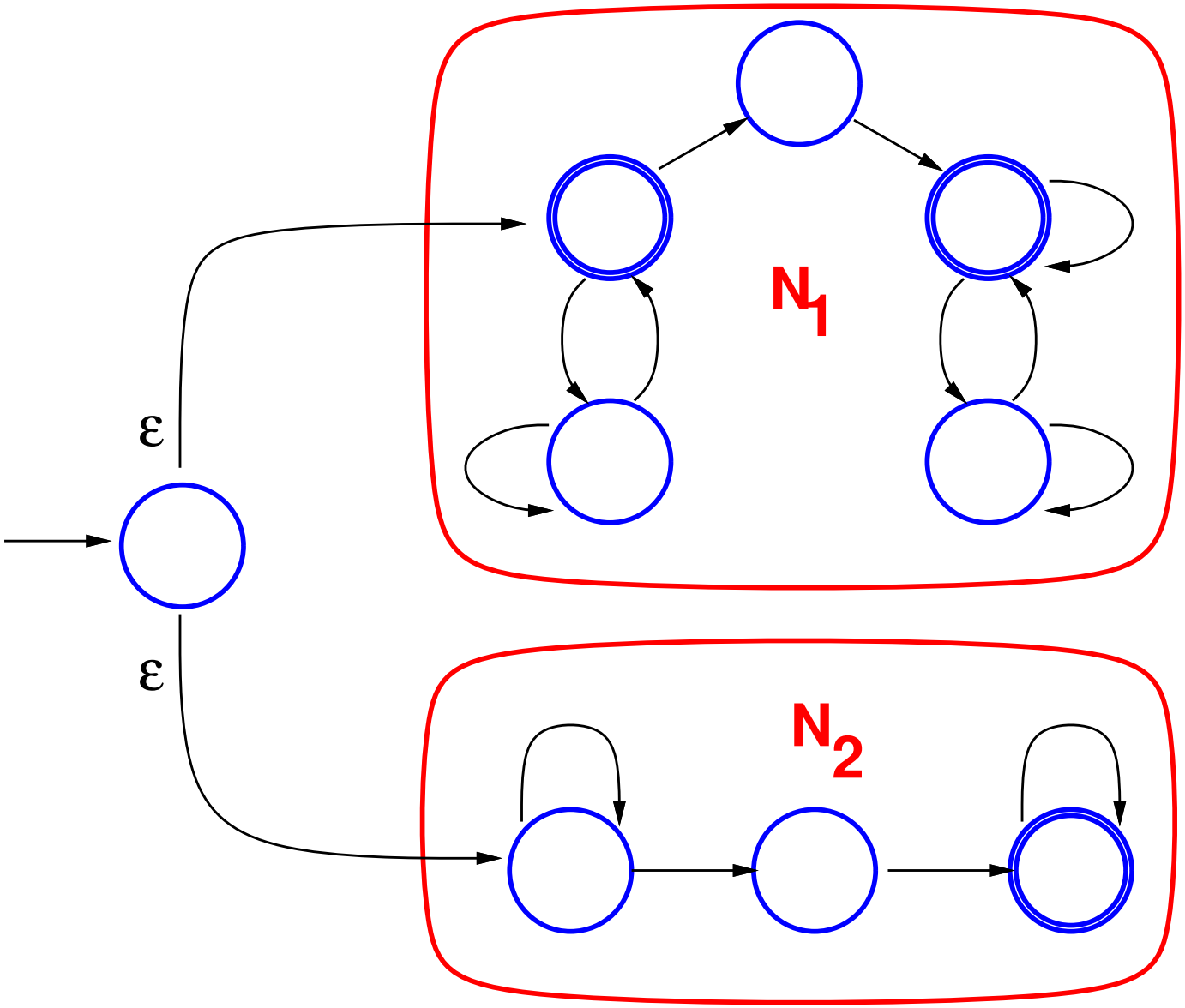
This is an alternative way of characterizing regular languages.

We will now use the equivalence to show that regular languages are closed under the regular operations (union, concatenation, star).

Closure under union (alternative proof)



Closure under union, cont.



Closure under union, cont.

- NFA $N_1 = (Q_1, \Sigma, \delta_1, S_1, F_1)$ accept $\mathcal{L}_1$, and
- NFA $N_2 = (Q_2, \Sigma, \delta_2, S_2, F_2)$ accept $\mathcal{L}_2$.

wlg. $Q_1 \cap Q_2 = \varnothing$.(?)

Define NFA $N = (Q = \{q_0\} \cup Q_1 \cup Q_2, \Sigma, \delta, S = \{q_0\}, F = F_1 \cup F_2)$,

for $\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ S_1 \cup S_2 & q = q_0 \text{ and } a = \varepsilon \end{cases}$

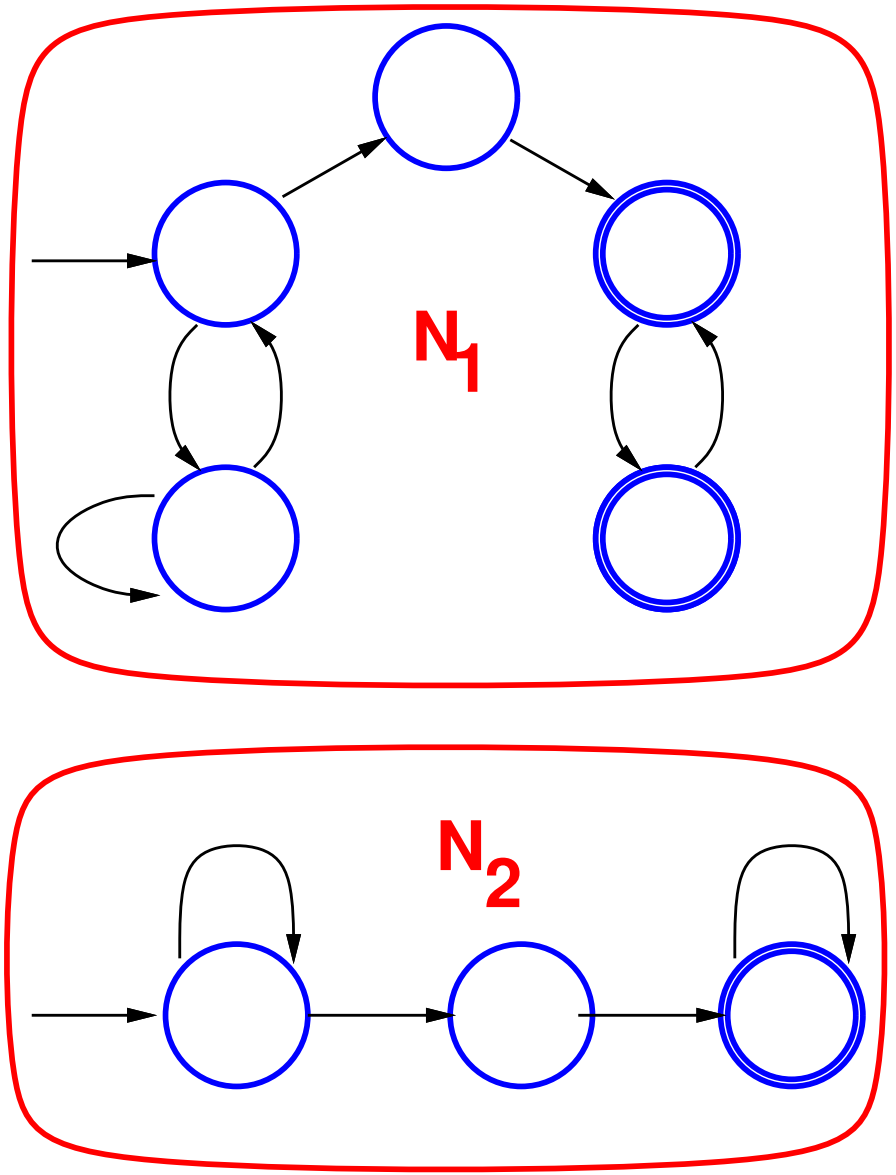
Alternatively, let $S = S_1 \cup S_2$ and omit the last line of δ.

Claim 20

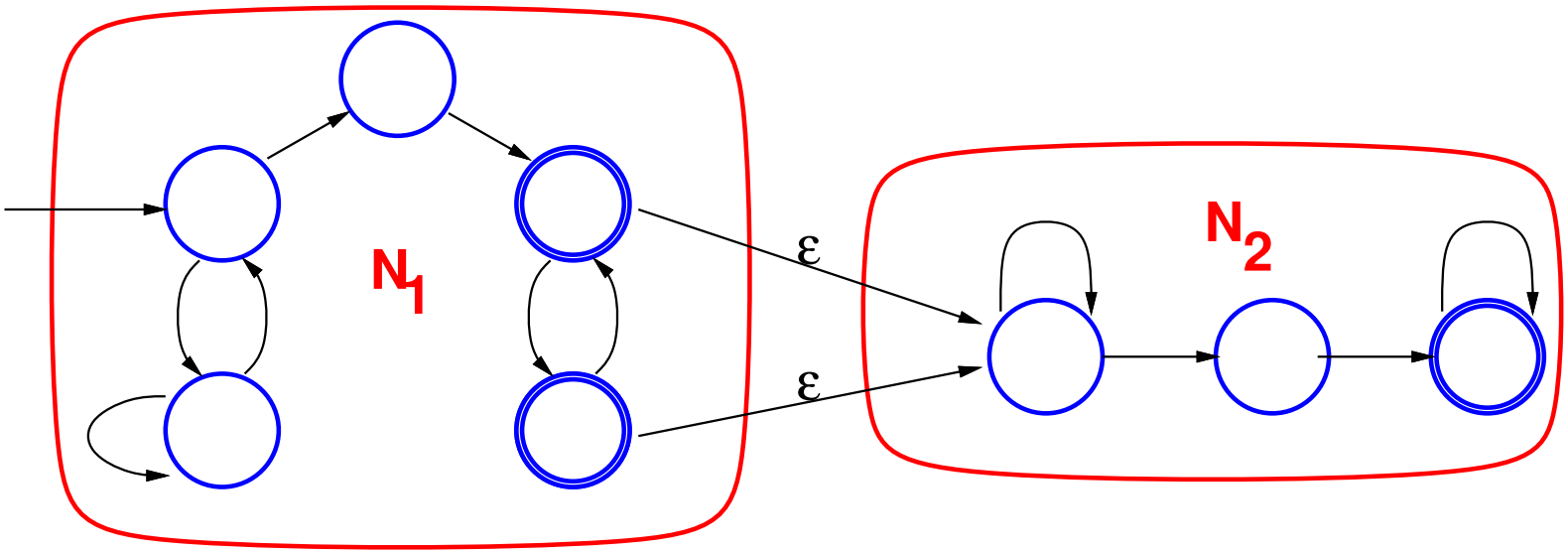
$\mathcal{L}(N) = \mathcal{L}(N_1) \cup \mathcal{L}(N_2)$.

Proof: Easily follow by the the alternative definition of acceptance by NFA, DIY...

Closure under concatenation



Closure under concatenation, cont.



Remark: Final states are exactly those of N_2.

Closure under concatenation, cont.

- NFA $N_1 = (Q_1, \Sigma, \delta_1, S_1, F_1)$ accept $\mathcal{L}_1$
- NFA $N_2 = (Q_2, \Sigma, \delta_2, S_2, F_2)$ accept $\mathcal{L}_2$

Define NFA $N = (Q_1 \cup Q_2, \Sigma, \delta, S_1, F_2)$:

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \wedge a \neq \varepsilon \\ \delta_1(q,a) & (q \in Q_1 \setminus F_1) \wedge a = \varepsilon \\ \delta_1(q,a) \cup S_2 & q \in F_1 \wedge a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

Claim 21

$\mathcal{L}(N) = \mathcal{L}_1 || \mathcal{L}_2$.

Proof: Need to prove $w \in \mathcal{L}(N) \iff w \in \mathcal{L}_1 || \mathcal{L}_2$.

Proving $w \in \mathcal{L}(N) \iff \mathcal{L}_1 \| \mathcal{L}_2$

Let $w \in \mathcal{L}_1 \| \mathcal{L}_2$:

$\Longrightarrow$ $\exists w^1 \in \mathcal{L}_1$ and $w^2 \in \mathcal{L}_2$, s.t. $w = w^1 w^2$

$\Longrightarrow$ $\exists\ r_0^1, \ldots r_{|w^1|}^1$ and $r_0^2, \ldots r_{|w^2|}^2$, such that for both $j \in \{1,2\}$:

(1) $r_0^j \in S_j$ (2) $r_{|w^j|}^j \in F_j$ (3) $\forall\ 0 \le i < \left|w^j\right|$: $r_{i+1}^j \in \delta_j(r_i^j, w_{i+1}^j)$.

$\Longrightarrow$ (details...) $r_0^1, \ldots r_{|w^1|}^1, r_0^2, \ldots r_{|w^2|}^2$ proves that $r_{|w^2|}^2 \in \hat{\delta}(S_1, w)$

$\Longrightarrow$ $w \in \mathcal{L}(N)$

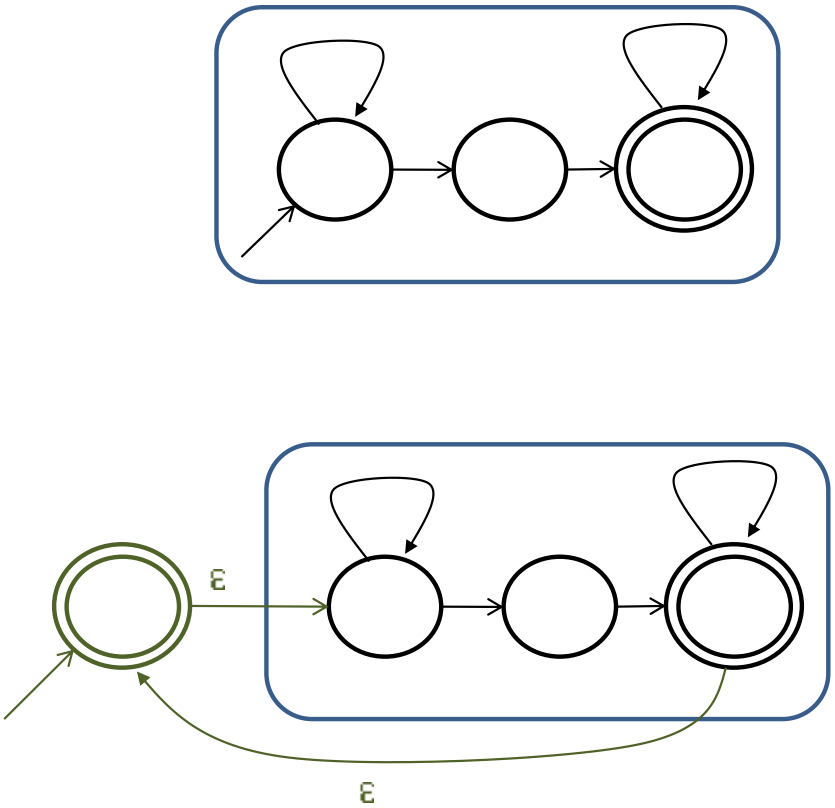
Let $w \in \mathcal{L}(N)$:

$\Longrightarrow$ $\exists\ w' = a_1 a_2 \ldots a_k \in (\Sigma_\varepsilon)^k$ and $r_0 \ldots r_k$ such that:
(1) $w = d(w')$ (2) $r_0 \in S_1$ (3) $r_k \in F_2$ (4) $\forall\ 0 \le i < k$: $r_{i+1} \in \delta(r_i, a_{i+1})$.

- Let j be the last index such that $r_j \in Q_1$
- By construction (details...) (1) $a_{j+1} = \varepsilon$ (2) $r_0 \ldots r_j$ proves that $d(a_1, \ldots, a_j) \in \mathcal{L}_1$ (3) $r_{j+1} \ldots r_k$ proves that $d(a_{j+2}, \ldots, a_k) \in \mathcal{L}_2$

$\Longrightarrow$ $w = d(w' = a_1, \ldots, a_j, \varepsilon, a_{j+2}, \ldots, a_k) = d(a_1, \ldots, a_j) d(a_{j+2}, \ldots, a_k) \in \mathcal{L}_1 \| \mathcal{L}_2$ $\square$

Closure under star



Closure under star, cont.

Let $N = (Q, \Sigma, \delta, S, F)$ accepting $\mathcal{L}$, assuming wlg. that $q_0 \notin Q$.

Define $N' = (Q' = Q \cup \{q_0\}, \Sigma, \delta', S' = \{q_0\}, F' = \{q_0\})$:

$$\delta'(q,a) = \begin{cases} \delta(q,a) & q \in Q \wedge a \neq \varepsilon \\ \delta(q,\varepsilon) & q \notin F \wedge a = \varepsilon \\ \delta(q,\varepsilon) \cup \{q_0\} & q \in F \wedge a = \varepsilon \\ S & q = q_0 \wedge a = \varepsilon \end{cases}$$

Claim 22

$\mathcal{L}(N') = \mathcal{L}(N)^*$.

Proof: DIY

Summary

- Regular languages are closed under
  - union
  - concatenation
  - star
- Nondeterministic finite automata
  - are equivalent to deterministic finite automata
  - but much easier to use in some proofs and constructions.

Part III

Regular Expressions

Regular expressions

Notation for building up languages by describing them as expressions, e.g., $(0 \cup 1)0^*$.

- 0 and 1 are shorthand for the set (languages) $\{0\}$ and $\{1\}$
- so $0 \cup 1 = \{0, 1\}$.
- 0^* is shorthand for $\{0\}^*$.
- Concatenation, is implicit. So 0^*10^* stands for $\{w \in \{0,1\}\colon\ w$ havs exactly a single $1\}$.
- Just like in arithmetic, operations have precedence:
  - star first
  - concatenation next
  - union last
  - parentheses used to change default order i.e., $ab^* \neq (ab)^*$

Q.: What does $(0 \cup 1)0^*$ stand for?

Remark: Regular expressions are often used in text editors or shell scripts.

Regular expressions – formal definition

Definition 23

A string R is a regular expression over alphabet Σ, if R is of form

- a for some $a \in \Sigma$
- ε
- $\varnothing$
- $(R_1 \cup R_2)$ for regular expressions R_1 and R_2
- $(R_1 || R_2)$ for regular expressions R_1 and R_2
- (R_1^*) for regular expression R_1

$R(\Sigma)$ denotes all (finite) regular expression over Σ.

Parenthesis and $||$ are omitted when their role is clear from the context.

Formal definition, cont.

Definition 24

The language $\mathcal{L}(R)$ of regular expression R, is defined by

R	$\mathcal{L}(R)$
a	$\{a\}$
ε	$\{\varepsilon\}$
$\emptyset$	$\emptyset$
$(R_1 \cup R_2)$	$\mathcal{L}(R_1) \cup \mathcal{L}(R_2)$
$(R_1 R_2)$	$\mathcal{L}(R_1) \| \mathcal{L}(R_2)$
(R_1^*)	$\mathcal{L}(R_1)^*$

Isn't this definition circular?

Examples of regular expressions

For $\Sigma = \{0, 1\}$, write regular expression for the following languages:

- The third letter from the end is 1

  $(0 \cup 1)^* 1 (0 \cup 1)^2$

- The number of 1's is even

  $(0 \cup 10^* 1)^*$

- The number of 1's is odd

  $(0 \cup 10^* 1)^* 10^*$

Part IV

Regular Expressions and Regular Languages

A remarkable fact

Theorem 25

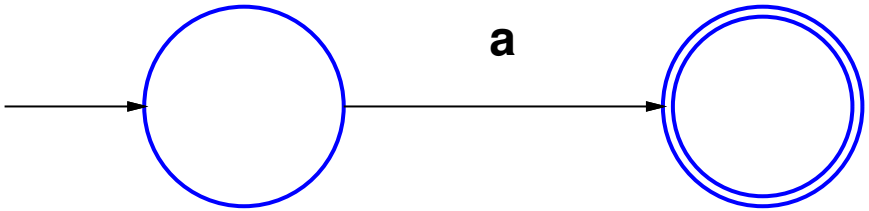
A language is described by a regular expression iff it is regular.

$\Longleftarrow$: Given a regular language, construct a regular expression describing it.

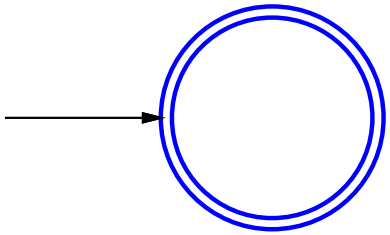
$\Longrightarrow$: Given a regular expression, construct an NFA accepting its language.

Given RE R, build NFA Accepting it ($\Longrightarrow$)

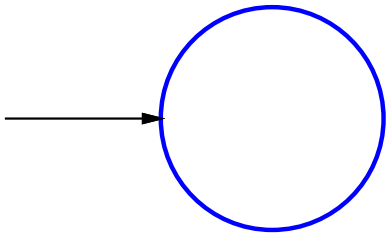
1. $R = a$, for some $a \in \Sigma$



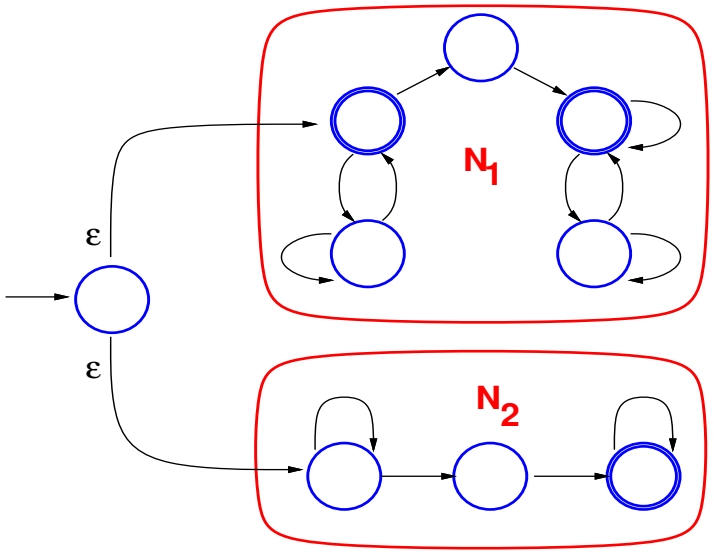
2. $R = \varepsilon$



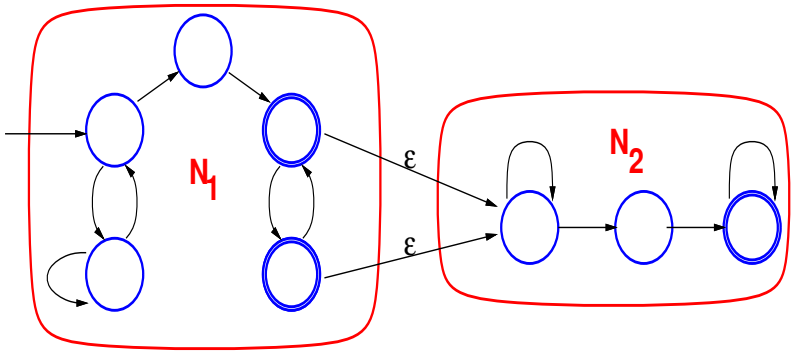
3. $R = \varnothing$



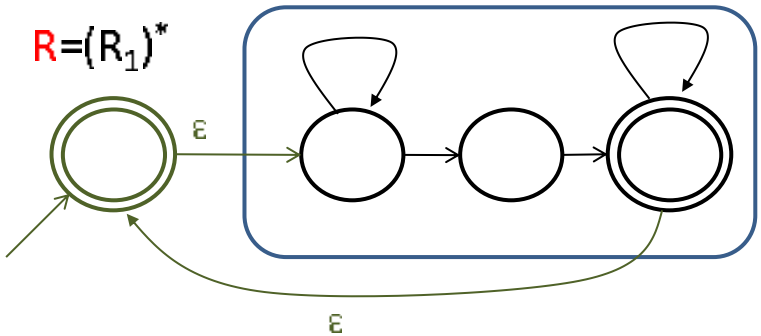
Given R, Build NFA Accepting It ($\Longrightarrow$), cont.



$R = (R_1 \cup R_2)$

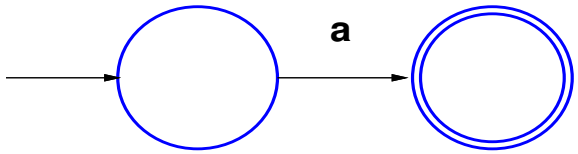


$R = (R_1 \| R_2)$

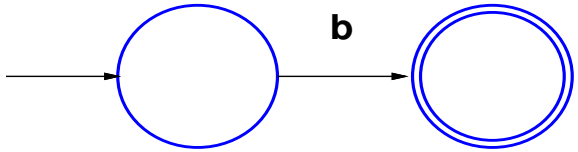


Examples

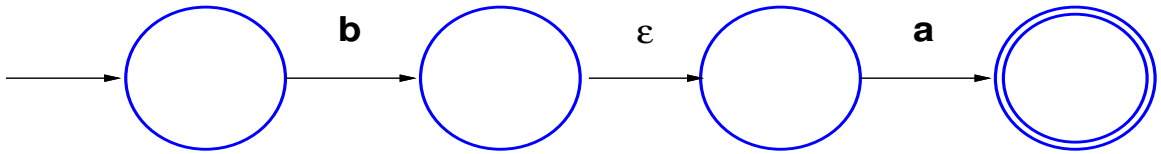
a



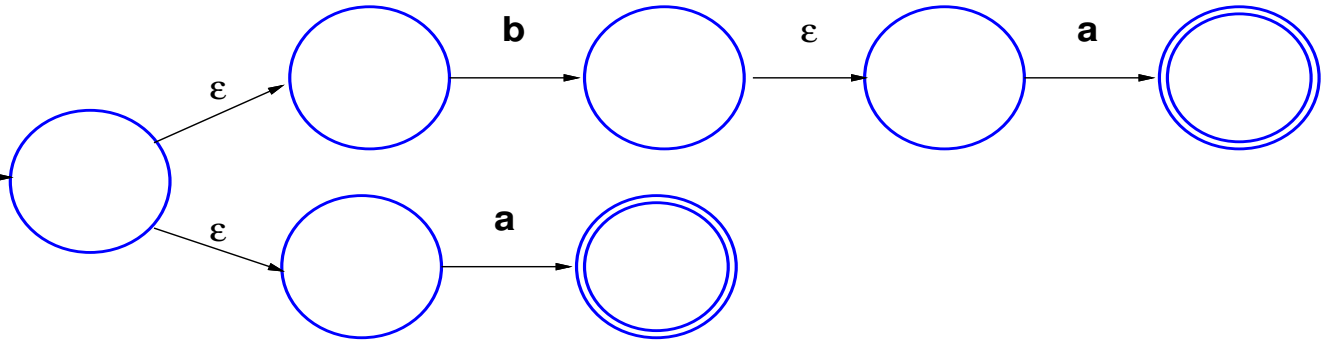
b



ba



ba ∪ a



Formal proof by induction on the length of the regular expression

Regular expression from a DFA A ($\Longleftarrow$)

Define

$$q \underset{S}{\rightsquigarrow} r = \left\{ a_1 \dots a_n \in \Sigma^* : q \xrightarrow{a_1} s_1 \longrightarrow \cdots \longrightarrow s_{n-1} \xrightarrow{a_n} r,\ s_k \in S \right\}$$

These are all the words traversed going from state q to state r via states in $S \subseteq Q$ only.

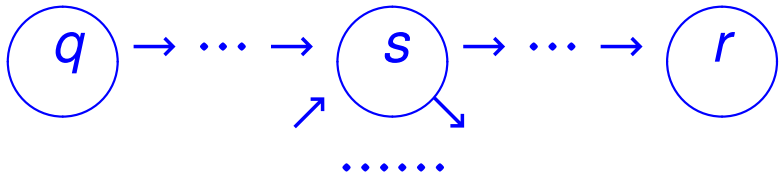
Construct the following regular expressions for $q \underset{S}{\rightsquigarrow} r$, recursively:

$$q \underset{\varnothing}{\rightsquigarrow} r = \{ a \in \Sigma : q \xrightarrow{a} r \} \cup \begin{cases} \varepsilon & q = r \\ \varnothing & q \neq r \end{cases}$$

$$q \underset{S \cup \{s\}}{\rightsquigarrow} r = q \underset{S}{\rightsquigarrow} r \cup \left[q \underset{S}{\rightsquigarrow} s \right] \left[s \underset{S}{\rightsquigarrow} s \right]^* \left[s \underset{S}{\rightsquigarrow} r \right]$$

Regular expression from a DFA A ($\Longleftarrow$)

$$q \underset{S\cup\{s\}}{\rightsquigarrow} r = q \underset{S}{\rightsquigarrow} r \cup \left[q \underset{S}{\rightsquigarrow} s\right]\left[s \underset{S}{\rightsquigarrow} s\right]^{*}\left[s \underset{S}{\rightsquigarrow} r\right]$$



Regular expression from a DFA A ($\Longleftarrow$)

If:

$$q \underset{S\cup\{s\}}{\rightsquigarrow} r = q \underset{S}{\rightsquigarrow} r \cup \left[q \underset{S}{\rightsquigarrow} s\right]\left[s \underset{S}{\rightsquigarrow} s\right]^{*}\left[s \underset{S}{\rightsquigarrow} r\right]$$

Then:

$$L(A) = q_0 \underset{Q}{\rightsquigarrow} f_1 \cup \cdots \cup q_0 \underset{Q}{\rightsquigarrow} f_n$$

for $F = \{f_1, \ldots f_n\}$.

Summary

- NonDeterministic Automata (with ε-moves)
  - Equivalence to DFA
- Closure properties
  - union
  - concatenation
  - star
- Regular expressions.
  - Equivalence to DFA